MATH 0290: Homework 7

Spring 2014

Due at the beginning of class on Wednesday, Mar. 5

You must show all your work to receive full credit. You are encouraged to discuss the homework with other students, but you must write up your own solutions. If you have any questions about the homework, please contact me in person or at `alk92@pitt.edu`.

Problem 1

(a) Plot the following functions, then write them in terms of Heaviside step functions.

(i)

$$f(t) = \begin{cases} \sin(t), & t \in [0, 2\pi) \\ t - 2\pi, & t \geq 2\pi \\ 0 \text{ otherwise} \end{cases}$$

(ii)

$$f(t) = \begin{cases} 1 - e^{-(t-1)}, & t \in [1, 2) \\ (1 - \frac{1}{e})e^{-(t-2)}, & t \geq 2 \\ 0 \text{ otherwise} \end{cases}$$

(b) Evaluate each of these integrals. Make sure to check the bounds of the integrals and compare them with the δ functions' arguments.

(i)

$$\int_0^{10} e^{-3t} \sin(2t)\delta(t-5)dt$$

(ii)

$$\int_2^4 \frac{1}{t^4+3}\delta(t-1)dt$$

(iii)

$$\int_3^7 \frac{\cos t}{1-t^3}\left(\delta(t-5)+\delta(t-8)\right) dt$$

Problem 2

Consider the following ODE:

$$y'' + 4y = f(t); \;\; y(0) = 0, \; y'(0) = 0.$$

Suppose the forcing is given by:

$$f(t) = \begin{cases} t-2, \; t \geq 2 \\ 0 \text{ otherwise} \end{cases}$$

Find $y(t)$ for $t > 0$ and plot it as a function of time.

Problem 3

(a) Find the convolution of the following functions, assuming that they are defined for positive t (recall that under this assumption, $f * g = \int_0^t f(u)g(t-u)du$). You may find it helpful to draw the overlap for a few values of t like we did in class, to get some intuition for what the integral looks like.

(i) $f(t) = e^{3t}, g(t) = e^{-2t}$.

(ii) $f(t) = t, g(t) = e^{-t}$.

(b) Use the proposition about the Laplace transform of $f(t-c)H(t-c)$ to find $y(t) = \mathcal{L}^{-1}[Y(s)]$.

(i) $Y(s) = e^{-3s}\frac{2s+1}{s^2+1}$

(ii) $Y(s) = e^{-2s}\frac{2}{(s+3)^2}$

Problem 4

Consider the following ODE:

$$y'' + 2y' + 10y = f(t); \;\; y(0) = 1, \; y'(0) = 1$$

(a) Find the input-free solution $y_i(t)$ (recall that this is the solution with $f(t)$ set to 0). You do not need to use Laplace transforms unless you want to.

(b) Now find the impulse response function $e(t)$.

(c) Suppose $f(t) = 2\delta(t-1) + \delta(t-3)$. Using the impulse response function, find the state-free solution $y_s(t)$ (recall this is the solution with the initial conditions set to zero).

(d) Using (a) and (c), write the full solution $y(t)$ to the ODE given the $f(t)$ defined above.

Problem 5

Consider the following ODE:

$$my'' + by' + cy = 0; \;\; y(0) = y_0, \; y'(0) = v_0$$

Recall that the impulse response function for this ODE is defined by:

$$me'' + be' + ce = \delta(t); \;\; e(0) = 0, \; e'(0) = 0$$

Take the Laplace transform of both of these equations and use the result to answer the following question: What should y_0 and v_0 be so that $y(t)$ and $e(t)$ are the same function?

Your answer should make sense in light of our discussion about impulses being related to changes in momentum.

Problem 6

Suppose $f(t)$ is periodic with period 1, and $f_T(t) = t$. In this case, $f(t)$ is often called a sawtooth function.

(a) Plot $f(t)$.

(b) Find the Laplace transform of $f(t)$.