MATH 0290: Homework 6

Spring 2014

Due at the beginning of class on Wednesday, Feb. 26

You must show all your work to receive full credit. You are encouraged to discuss the homework with other students, but you must write up your own solutions. If you have any questions about the homework, please contact me in person or at `alk92@pitt.edu`.

Problem 1 (PBA 5.2 # 42)

Here is another way to find $\mathcal{L}[\cos \omega t]$.

(a) Using only the definition of the Laplace transform, show that $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$. Note: this means that you need to actually do the integral, not look up the identity in a table.

(b) If $\mathcal{L}[f(t)] = F(s)$, use the definition of the Laplace transform to show that:

$$\mathcal{L}[f(at)] = \frac{1}{a}F(s/a)$$

Given this show that $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$.

(c) Use the proposition about $\mathcal{L}[f'(t)]$ and the answer to the previous part of the problem to find $\mathcal{L}[\cos \omega t]$.

Problem 2

(a) (PBA 5.3 # 2) Find $y(t)$ if $Y(s) = \frac{2}{3-5s}$.

(b) (PBA 5.3 # 10) Find $y(t)$ if $Y(s) = \frac{1}{s^5} - \frac{5-9s}{s^2+100}$.

(c) (PBA 5.3 # 24) Find $y(t)$ if $Y(s) = \frac{7-s}{s^2+s-2}$. Show your work for the partial fraction decomposition.

Problem 3

Let's return to the damped, undriven harmonic oscillator:

$$y'' + 2cy' + \omega_0^2 y = 0; \quad y(0) = y_0, y'(0) = v_0$$

(a) Compute the Laplace transform and solve for $Y = \mathcal{L}[y]$.

(b) Assume that $\omega_0 = c$ (critical damping). Use the inverse Laplace transform and your answer to part 1 to show that $y = y_0 e^{-ct} + (v_0 + cy_0)te^{-ct}$.

(c) Now consider the driven case: $y'' + 2cy' + \omega_0^2 y = A\cos\omega t; \quad y(0) = y_0, y'(0) = v_0$. Solve for $Y = \mathcal{L}[y]$. You only need to solve for $Y(s)$, not $y(t)$.

Problem 4

(a) Compute the Laplace transform of:

$$f(t) = \begin{cases} t, \ t \in [0,1) \\ 0, \ \text{otherwise} \end{cases}$$

(b) Compute the Laplace transform of:

$$f(t) = \begin{cases} 1, \ t \in [0,1) \\ -1, \ t \in [1,2) \\ 0, \ \text{otherwise} \end{cases}$$

Problem 5

Consider the following ODE:

$$y'' + 4y = \cos(2t); \quad y(0) = 0, y'(0) = 2$$

(a) Find $Y_i(s)$, the Laplace transform of the *input-free* problem (no forcing).

(b) Find $Y_s(s)$, the Laplace transform of the *state-free* problem ($y(0) = y'(0) = 0$).

(c) Find the inverse of each of these and use it to compute the particular solution $y(t)$. You can use the following identity:

$$\mathcal{L}[t\sin at] = \frac{2as}{(a^2+s^2)^2}$$