MATH 0290: Homework 5

Spring 2014

Due at the beginning of class on Wednesday, Feb. 19

You must show all your work to receive full credit. You are encouraged to discuss the homework with other students, but you must write up your own solutions. If you have any questions about the homework, please contact me in person or at `alk92@pitt.edu`.

Problem 1: RLC Circuit

Problems 1 and 2 are not difficult, but you need to plot some things in MATLAB. I suggest you start them first in case you have problems with your code.

In this problem, we will consider the dynamics of a circuit consisting of a resistor ($R = 400\ \Omega$; Ω represents *Ohms*), a capacitor ($C = 10^{-4}$ F; F represents *Farads*), and an inductor ($L = 25$ H; H represents *Henries*) in series, forced by $F(t) = B\cos(\omega t)$. The equation for the current running through this circuit $I(t)$ is:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I/C = B\cos(\omega t). \tag{1}$$

We will derive this equation in class on Monday.

(A) Rewrite the equation in our standard form for forced harmonic motion (determine c and ω_0 in terms of R, L, C):

$$I'' + 2cI' + \omega_0^2 I = A\cos(\omega t). \tag{2}$$

Write out the numerical value of c and ω_0 from the given parameters. Include the units. Note that 1 H $\cdot$1 F = 1 s^2 and $1\Omega/1\text{H}$ = 1/s.

(B) In class, we derived the *transfer function* $H(\omega) = G(\omega)e^{i\phi(\omega)}$ for equation (2). We found that the *gain* was:

$$G(\omega) = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4c^2\omega^2}} \tag{3}$$

The *phase lag* was given by:

$$\tan(\phi(\omega)) = \frac{-2c\omega}{\omega_0^2 - \omega^2} \tag{4}$$

We're going to plot these quantities in MATLAB as a function of ω, creating what is known in engineering as a *Bode plot*. It will tell us everything we need to know about the RLC circuit. If we give an input of size A, we will be able to find the magnitude of the output $A \cdot G(\omega)$ and the phase lag $\phi(\omega)$. For this subproblem, all you need to do is hand in the plots you produce in (iv).

(i) Define your vector of ω going from $\omega = 10^{-2}$ Hz to $\omega = 10^4$ Hz: `w = 0.01:0.01:10000;`

(ii) Calculate $G(\omega)$ from this vector. Hint: In MATLAB, when you want to perform an operation other than addition or subtraction on each element of a vector individually, you need to precede the operation by the period symbol. So, for example, if you wanted to compute $f(\omega) = 1/(1+\omega^2)$ for a vector ω, the MATLAB code would be:

```
f = 1./(1 + w.^2);
```

I have written up some MATLAB code that introduces how to deal with vectors: `plotvec.m`.

(iii) Calculate $\phi(\omega)$. There is one issue: arctangent is typically defined on $(-\pi/2, \pi/2)$, but we want $\phi \in (-\pi, 0)$ so that it is always negative. This is because positive phase lag would mean that the system would be responding to an input before the input happened. To get this to work, use the command `atan2(x,y)`, where `x` is the numerator of equation (4) and `y` is the denominator.

(iv) Now plot your results. The plot of $G(\omega)$ should be on a log-log axis. The command `loglog(w,G)` will accomplish this.

For $\phi(\omega)$, the x-axis (frequency) should be logarithmic but the y-axis (phase lag) should be linear. The command `semilogx(w,phi)` will accomplish this. Make sure to label all your axes. I will not take late or emailed plots.

As a check, you should find that for high ω, $G(\omega)$ decreases proportional to $1/\omega^2$. On a log-log plot, this corresponds to a line with slope -2. You should also find that $\phi(\omega)$ is between $-\pi$ and 0 and is decreasing.

Problem 2: Forced harmonic motion, continued

You will notice that there is a peak in $G(\omega)$. This is the forcing frequency that elicits the highest amplitude response.

(A) Which element of the circuit (the resistor, capacitor, or inductor), causes damping? If damping were *not* present, at what ω would this peak occur?

(B) You will notice that the actual peak is not exactly at the value of ω found in (A). Using equation (3), find the ω that maximizes $G(\omega)$. You should find that it matches the location of the peak on your plot. Hint: If $A(\omega) = 1/\sqrt{f(\omega)}$ with $f > 0$, then $A(\omega)$ has a maximum when $f(\omega)$ has a minimum.

(C) Multiply the value of the circuit element that you found in (A) (R, C, or L) by 0.1 and plot $G(\omega)$ again. This will show what happens to the gain if damping is reduced by a factor of 10. Is the peak sharper or less sharp than in the previous problem?

Problem 3: Undetermined coefficients

Find the *general* solution to the given ODEs using the method of undetermined coefficients (to do this you will first need to find the general solution to the homogeneous equation y_h, then find the particular solution y_p, and use these to determine y).

(A) $y'' + 7y' + 10y = -4\sin 3t$

(B) $y'' - 3y' - 10y = 3e^{-2t}$. Note: the forcing term is a solution of the homogeneous equation here.

Problem 4: Variation of parameters

Find the *general* solution to the given ODEs using the method of variation of parameters (to do this you will first need to find the general solution to the homogeneous equation y_h, then find the particular solution y_p, and use these to determine y).

(A) $y'' + 4y = \sec(2t)$

(B) $t^2y'' + 3ty' + y = 1/t$, given that $y_1 = 1/t$ and $y_2 = \ln(t)/t$ are solutions to the homogeneous equation.