MATH 0290: Homework 3

Spring 2014

Due at the beginning of class on Wednesday, Jan. 29

You must show all your work to receive full credit. You are encouraged to discuss the homework with other students, but you must write up your own solutions. If you have any questions about the homework, please contact me in person or at `alk92@pitt.edu`.

Problem 1 (PBA 2.7 # 10)

Show that $y(t) = 0$ and $y(t) = (1/16)t^4$ are both solutions of the initial value problem $y' = t\sqrt{y}$, where $y(0) = 0$. Explain why this does not contradict the uniqueness theorem discussed in class (Theorem 7.16 in the book).

Problem 2

(A) (PBA 2.2 #14) Find the particular solution of the initial value problem. Indicate the interval of existence.

$$y' = -\frac{2t(1+y^2)}{y}; \;\; y(0) = 1$$

(B) (PBA 2.4 #10) Find the general solution to the following equation:

$$y' = my + c_1 e^{mx}$$

where m and c_1 are real constants.

(C) (PBA 2.4 #16) Find the solution of the initial value problem:

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}; \;\; y(1) = 0$$

(D) (PBA 2.6 #12) Find the general solution:

$$\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = 0$$

Problem 3

(A) Suppose you have an account that offers a 5% interest rate. You want to invest P_0 dollars at time $t = 0$ and withdraw $500 per year, for 20 years. What should P_0 be so that at $t = 20$ years you have $75,000 in the account?

For this problem and the next you can assume, as we did in class, that the interest is compounded continuously and that your withdrawals are also done continuously. If you have trouble setting the problem up, refer to the second example in the personal finance section that we did in lecture.

(B) Suppose I have an account with an interest rate r. It is initially empty, but I will contribute $1,000 to it every year for 20 years. What does r need to be so that at the end of 20 years, I have $30,000 in the account? It is okay to use a graphical solution (for example, plotting $P(20 \text{ years})$ as a function of r) to solve for r.

Problem 4

In class, we solved the logistic model of population growth. Like we did on the first homework, let's see what happens when we make the rate of births and deaths oscillate.

Consider the following ODE for the size of a population $P(t)$:

$$P' = P(1 - P/4)(1 + \sin 5t)$$

This is the same as the logistic growth equation, except we have an extra term $1 + \sin 5t$.

(A) There are two solutions with $P' = 0$, which are identical to the ones we would get for standard logistic growth. What are they? Given these solutions and the uniqueness theorem, what is the maximum value a solution $P(t)$ can obtain if $P(0) = 2$?

(B) Rather than solve the equation, let's use `DFIELD`. Plot the direction field for the above equation using `DFIELD` (the link to download it is on the course website). Set the axes to the ranges $t \in (0, 10)$ and $P \in (0, 5)$. Click on the direction field so that `DFIELD` plots one solution corresponding to each of the constant solutions from part (A), and one additional solution corresponding to an initial condition in between these two constant solutions. Turn in your plot. I won't take late plots this time.

Problem 5

Suppose we have an ODE $y' = f(t, y)$, and f is continuously differentiable. Also suppose that $y_1(t) = t^2$ is one of the particular solutions to the ODE. Give an upper bound to $y(5)$, where $y(t)$ is the solution of the initial value problem:

$$y' = f(t, y); \quad y(2) = 3$$

Hint: Use the intuition from our discussion about uniqueness of solutions.