MATH 0290: Quick reference for first order ODEs

Spring 2014

January 22, 2014

This is not a comprehensive study guide for all the material we have discussed on first order ODEs. It is meant to be a quick reference for notation and methods of solution for the three classes of first order ODEs that we know how to solve analytically.

1 Definitions

A *first order ordinary differential equation* (ODE) is an equation involving an unknown function $y(t)$, an independent variable t, and the first derivative of y with respect to t, $\frac{dy}{dt} = y'$.

$$0 = g(t, y, y')$$

If we solve for y' to put the above equation in *normal form*, we get the more familiar equation:

$$y' = f(t, y)$$

A *solution* of the above ODE is a function $y(t)$ that satisfies the above expression. An *initial value problem* is an ODE along with an *initial condition*:

$$y' = f(t, y); \;\; y(t_0) = y_0$$

There are two types of solutions to ODEs. The first are *general solutions*, which involve unknown parameters (such as C). For example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$. General solutions give all the possible solutions to a given ODE.

Particular solutions are solutions that satisfy the initial condition as well as the ODE for an initial value problem. For example, $y(t) = 2e^{at}$ satisfies the initial value problem $y' = ay; \;\; y(0) = 2$.

We also differentiate between *explicit solutions*, which solve for $y(t)$ as a function of t, and *implicit solutions*, in which we don't fully solve for $y(t)$. $y(t) = t$ is explicit; $y^2(t) + t^2 = 1$ is implicit.

2 Separable equations

A *separable equation* is one that can be written as:

$$g(y)dy = h(t)dt$$

That is, it is possible to *separate* the ys and the ts onto different sides of the equals sign. Here's how to get the general solution for a separable equation:

1. Separate the variables so that the equation is of the form above.
2. Integrate both sides: $\int g(y)dy = \int h(t)dt + C$. Don't forget the constant of integration.
3. Solve for y.

3 The simplest ODE

In class, we have continually encountered ODEs of the form $y' = ry$ for some constant r. This is a separable ODE. We can solve it using the technique above:

$$\begin{aligned} \frac{dy}{dt} &= ry \\ \frac{dy}{y} &= rdt \\ \int \frac{dy}{y} &= \int rdt \\ \ln y &= rt + C \\ y &= Ce^{rt} \end{aligned}$$

The fact that ODEs of this form (derivative proportional to the function itself) have solutions that are exponentials should start becoming familiar to you. You should also understand that $y(t) = y_0 e^{r(t-t_0)}$ is the particular solution given the initial condition $y(t_0) = y_0$. You should get a feeling for what these functions do for positive r (blow up) and negative r (decay to 0), starting from the initial condition y_0. Increasing the magnitude of r makes these blow ups or decays happen more rapidly.

4 Linear equations

A *linear equation* is one that can be written as:

$$y' - a(t)y = f(t)$$

If $f(t) = 0$, the equation is homogeneous; if not, it is inhomogeneous. Notice that homegenous equations are separable, so they are easy to solve. Below is the method of integrating factors for solving a linear ODE:

1. Put the ODE in the form above.

2. Determine the integrating factor: $u(t) = e^{-\int a(t)dt}$.

3. Multiply both sides of the equation by u: $u(t)y' - a(t)u(t)y = f(t)u(t)$.

4. Notice that this can be rewritten as $(uy)' = f(t)u(t)$.

5. Integrate both sides: $uy = \int f(t)u(t)dt + C$. Don't forget the constant of integration.

6. Solve for y: $y = \frac{1}{u(t)}\left(\int f(t)u(t)dt + C\right)$.

5 Exact equations

An *exact equation* is one that is of the form:

$$P(x, y)dx + Q(x, y)dy = 0$$

where $P(x, y) = \partial F/\partial x$ and $Q(x, y) = \partial F/\partial y$, for some function F. If an equation is exact, our job is to find F.

Knowing F is useful because the general solution to the exact equation is $F(x, y) = C$, where C is a constant. This is because $P(x, y)dx + Q(x, y)dy = 0$ can be written as:

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

which means that F is constant.

To test if an equation is exact, we can use the following observation:

$$\begin{aligned} \frac{\partial^2 F}{\partial y \partial x} &= \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \\ \frac{\partial P(x, y)}{\partial y} &= \frac{\partial Q(x, y)}{\partial x} \end{aligned}$$

The last line above tells us a "test" for exactness. If it's satisfied, we're guaranteed that our equation is exact. If we know our equation is exact, we can take the following steps to find the general solution:

1. Write $F(x, y) = \int (\partial F/\partial x)dx$.

2. Integrate to get $F(x, y) = \int P(x, y)dx + \Phi(y)$, where $\Phi(y)$ is an arbitrary function of y.

3. Do the same for Q. Write $F(x, y) = \int (\partial F/\partial y)dy$.

4. Integrate to get $F(x, y) = \int Q(x, y)dy + \Psi(x)$, where $\Psi(x)$ is an arbitrary function of x.

5. Determine what $\Phi(y)$ and $\Psi(x)$ are so that you know fully what $F(x, y)$ is.
6. Set $F(x, y) = C$. This is your general solution. It is often an implicit solution.