MATH 0290: Topics for exam 1

Spring 2014

April 8, 2014

This is a short summary of the topics that you are required to know for exam 1. The material goes from the first lecture through the beginning of the lecture on Monday, February 3. That means you need to know about linear, homogeneous equations with constant coefficients, but you don't need to know about the method of undetermined coefficients.

1 First third of the course

1.1 General ODE concepts

1. Definition of an ODE
2. The order of an ODE
3. What it means for a function to be a solution to an ODE
4. Initial value problems
5. General vs. particular solutions
6. Explicit vs. implicit solutions
7. Intervals of existence

1.2 First order ODEs

1. Separable ODEs
2. Linear ODEs
3. Exact ODEs
4. Direction fields
5. Uniqueness of solutions and what it means for solutions
6. Numerical solutions: Euler integration

1.3 Modeling

1. The Malthusian model of population growth
2. The logistic model of population growth
3. Personal finance: accounts with compounded interest, withdrawals, deposits

1.4 Second order ODEs

1. The form of initial value problems for second order ODEs
2. Linear combinations: $C_1y_1 + C_2y_2$
3. Linear dependence vs. linear independence
4. General form of solutions (in terms of linear combinations of the fundamental set of solutions)
5. How to solve second order equations with constant coefficients
6. Intuition for the three cases: real distinct roots, complex roots, and repeated roots
7. Intuition for the behavior of linear, homogeneous spring-mass systems: natural frequency, damping coefficient, spring constant, overdamped vs. underdamped vs. critically damped

2 Second third of the course

2.1 Second order ODEs

1. Second order equations with constant coefficients: $y(t) = C_1y_1(t) + C_2y_2(t)$.

   Real distinct, complex conjugate, and repeated roots of characteristic polynomial

2. The method of undetermined coefficients for equations of the form $y'' + py' + qy = f(t)$, where $f(t)$ takes a form that is repeated under differentiation. General solution: $y(t) = y_p + y_h$. Here y_p is the particular solution corresponding to the forcing, and y_h is the general solution to the homogeneous ODE (written above).

   How to deal with the case where $f(t)$ is a solution of the homogeneous ODE: multiply your guess for y_p by t and solve for it.

3. The method of variation of parameters for equations of the form $y'' + py' + qy = f(t)$. In this case, $y_p = v_1y_1 + v_2y_2$. The formulas for v_1 and v_2 will be provided (see the handout). General solution takes a similar form as above.

4. The method of variation of parameters for equations of the form $y'' + p(t)y' + q(t)y = f(t)$, assuming that $y_1(t)$ and $y_2(t)$ are given.

5. Harmonic motion

    Damped, underdamped, and critically damped cases

    Driven harmonic motion and the transfer function (transfer function is provided on the handout)

2.2 Laplace transforms

1. How to take the Laplace transform and the inverse Laplace transform

    Using the properties of the Laplace transform to find $\mathcal{L}[f]$ or $\mathcal{L}^{-1}[F]$ (table is included in the handout).

    You should be comfortable with partial fractions, completing the square, and in general the algebra required to put solutions in the correct form

2. Solving ODEs using the Laplace transform

    Direct approach: Find $Y(s)$ and invert it

    Splitting the solution into input-free (no forcing) and state-free ($y(0) = y'(0) = 0$) solutions

    Finding the impulse response function and writing the solution as a convolution: $e * f(t)$

3. Discontinuous forcing terms

    Writing discontinuous functions in terms of Heavisides

    Laplace transforms of Heavisides and their inverse (equation is provided on the handout)

    Periodic functions and their Laplace transforms (equation is provided on the handout)

    The δ-function. How to take integrals involving it, and when those integrals are nonzero (depending on the bounds of the integral). Taking Laplace transforms.

3 Final third of the course

3.1 Systems of equations

1. Definition of a system of ODEs

2. Linear algebra

    Finding eigenvalues using the characteristic equation

    Finding eigenvectors

    Trace and determinant of a matrix

3. Linear planar systems

    Form of the general solution (distinct eigenvalues): $\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$

    Form of the general solution (repeated eigenvalues): $\mathbf{x} = e^{\lambda t}\left[C_1\mathbf{v}_1 + C_2(\mathbf{v}_2 + t\mathbf{v}_1)\right]$

    Eigenvalues of 2×2 systems in terms of the trace and determinant

Subcases: nodal/spiral sources/sinks, lines of equilibria, centers, saddles

How to draw phase plane portraits for these given eigenvalues/eigenvectors

4. Stability

   Definition of a fixed/equilibrium point

   Meaning of stability of a fixed point

   One-dimensional systems: phase lines and calculating stability based on slope of $f(x)$

   Stability for linear systems (stable if real part of all eigenvalues ≤ 0, unstable if any > 0)

5. Nonlinear systems

   Linearization of a nonlinear system and the Jacobian

   Stability for nonlinear systems using the Jacobian (stable if real part of all eigenvalues < 0, unstable if any > 0, uncertain otherwise)

   Behavior near a fixed point

   Nullclines

   Definition of a limit cycle, positively invariant set, and limit set

   Generic possibilties for the limit set (fixed points or limit cycles)

   Drawing phase plane portraits

3.2 Fourier series

1. Definition of a Fourier series on $[-\pi, \pi]$ or $[-L, L]$
2. Calculating Fourier series for a given function
3. Using orthogonality relations for sine and cosine
4. Fourier series for even and odd functions